The Laplace-Beltrami Spectrum on Naturally Reductive Homogeneous Spaces

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The Laplacian

”Encode spectral information as symmetry information, i.e. representations!”

The Laplace-Beltrami Operator

Let (M, g) be a compact Riemannian manifold, $\dim(M) = m$.

Definition

The Laplace-Beltrami operator: $\Delta : \mathcal{C}^\infty(M, g) \to \mathcal{C}^\infty(M, g)$ is defined by $\Delta f := -\operatorname{div}\operatorname{grad} f$

- much geometry is encoded in eigenvalues $\eta \in \Sigma(M, g)$ and in eigenfunctions $f \in E_\eta$: $\Delta f = \eta f$

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The Laplacian encodes highly the geometry of the manifold:

$f : M \to N$ a diffeomorphism

$$\Delta_M(u \circ f) = (\Delta_N u) \circ f \quad \forall u \in \mathcal{C}^\infty(N) \quad \Leftrightarrow \quad f \text{ is an isometry}$$

$\rightsquigarrow$ isometries $f : M \to M$ reduce the complexity of the Laplacian!

The Laplacian on Homogeneous Spaces

Definition

If for any $p, q \in M$ there exists an isometry $f_{pq} : M \to M$ s.t. $f_{pq}(p) = q$, then (M, g) is called a homogeneous space.

- From now on: $G \subset \mathrm{Iso}(M, g)$ acts transitively, $M \cong G/K$ is a homogeneous space
- Laplacian commutes with isometries, $u \in \mathcal{C}^\infty(G/K)$

$\Rightarrow$ $\Delta(u)(aK) = \Delta(u(a\cdot))(eK)$

Know Δ on $G/K \Leftrightarrow$ know Δ in $eK \in G/K$!

Normal & naturally reductive Homogeneous spaces

$(G/K, g)$ is reductive: there exists a complement $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ such that $\mathrm{Ad}_K\, \mathfrak{m} \subset \mathfrak{m}$. We have $T_{eK}(G/K) \cong \mathfrak{m}$

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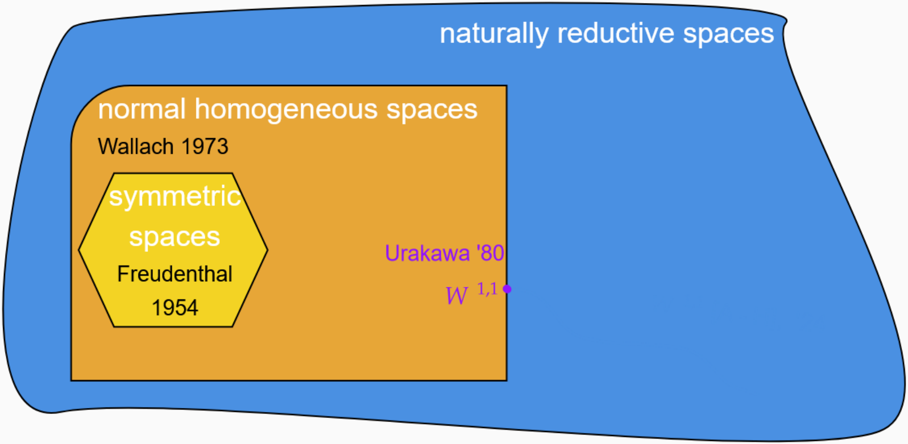
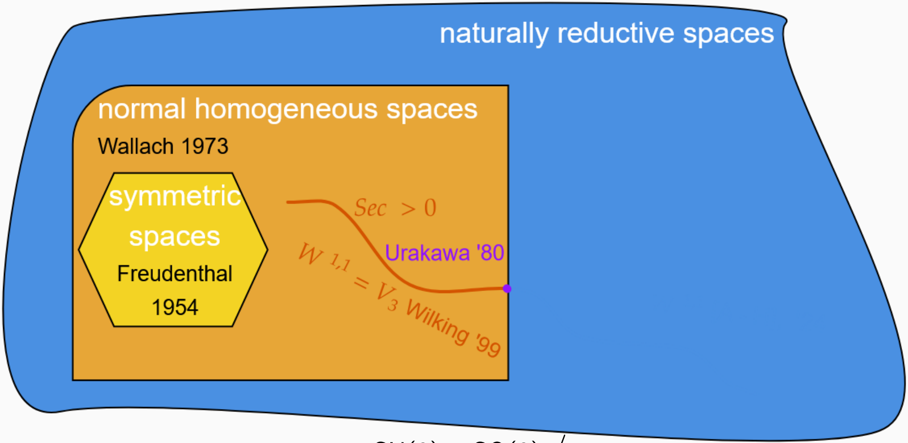
Last two definitions heavily depend on G/K (several options) and the $\mathrm{Ad}(G)$-invariant metric g of G!

The Laplace-Beltrami Spectrum



$W^{1,1} = \mathrm{SU}(3) / S^1$ "Aloff-Wallach manifold"

Wilking's normal homogeneous realization $V_3 = W^{1,1}$



naturally reductive spaces

normal homogeneous spaces

Wallach 1973

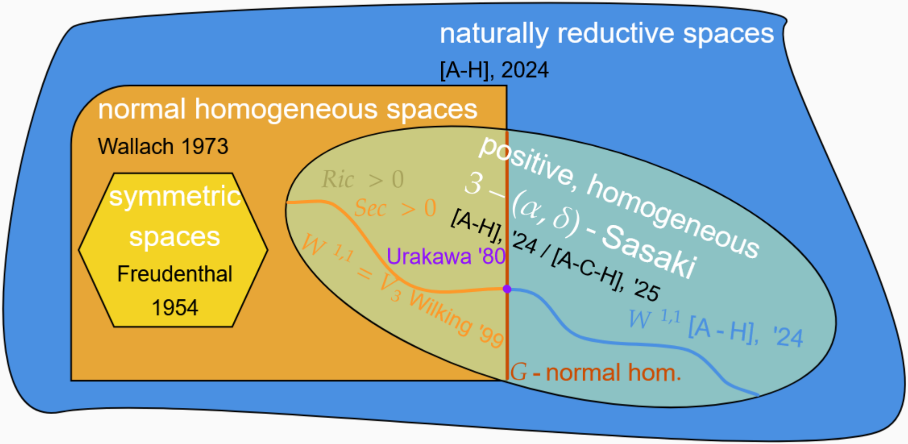
symmetric spaces

Freudenthal 1954

$$V_3 = \mathrm{SU}(3) \times \mathrm{SO}(3) \Big/ \mathrm{U}^{\bullet}(2),$$

(Wilking did not compute the spectrum)

Our contribution



naturally reductive spaces

[A-H], 2024

normal homogeneous spaces

Wallach 1973

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Freudenthal 1954

$Ric > 0$

$Sec > 0$

$W^{1,1} = V_3$ Wilking '99

Urakawa '80

positive, homogeneous $3-(\alpha, \delta)$ - Sasaki

[A-H], '24 / [A-C-H], '25

$W^{1,1}$ [A - H], '24

G - normal hom.

From Analysis to Representation Theory

- Let $(G/K, g)$ be naturally reductive
- The Laplacian on G/K can be identified with an algebraic object, the generalized Casimir element $\mathrm{C} \in \mathrm{U}(\mathfrak{g})$.
- The space of functions decomposes into irreducible representations of G (Peter-Weyl Theorem):

$$L^2(G/K) = \overline{\bigoplus_{\varrho \in \widehat{G}_K} V_\varrho \otimes V_\varrho^K}$$

- Only certain representations appear, the so-called K-spherical ones.

Definition: K-spherical Representations

A unitary, irred. rep. $(\varrho, V_\varrho) \in \widehat{G}$ is called *K-spherical* if it admits non-trivial K-invariant vectors, i.e.

$$V_\varrho^K = \{v \in V_\varrho \mid \varrho(K)v = v\}, \quad m(\varrho) = \dim(V_\varrho^K) \neq 0.$$

The set of K-spherical rep. is denoted by $\widehat{G}_K$ and $m(\varrho)$ is called the spherical multiplicity of ϱ.

- For naturally reductive spaces: On each such representation, the Casimir element C acts as a simple scalar $c_g(\varrho)$.

The Spectrum of Naturally Reductive Spaces

This algebraic framework yields a powerful formula for the spectrum.

Theorem (Wallach '73, A-H. '24)

The spectrum of a naturally reductive space $(G/K, g)$ is given by:

$$\Sigma(G/K, g) = \{c_g(\varrho) \mid \varrho \in \widehat{G}_K\},$$

where $c_g(\varrho) = g(\lambda + 2\rho, \lambda)$.
Here, λ is the highest weight of ϱ and $\rho = \frac{1}{2}\sum_{\mu \in R^+} \mu$.
The multiplicity of the eigenvalue $c_g(\varrho)$ is:

$$\mathrm{mult}(\varrho) = m(\varrho) \cdot \dim(V_\varrho)$$

This formula is our main tool, but it is only applicable if the space is naturally reductive.

Spectrum of Deformed Normal Homogeneous Spaces

”Want to consider families of metrics depending on deformation parameters, not just the restriction of Killing forms!”

Deformation of Normal Homogeneous Spaces

Our setting

- Let $(G/K, g)$ be normal homogeneous and $H \subset G$ a connected subgroup commuting with K, with $\mathfrak{k} \perp \mathfrak{h}$.
- We get a Riemannian submersion with totally geodesic fibers:

$$H \Big/ (H \cap K) \xrightarrow{\text{vertical } \mathcal{V}} G \,/\, K \xrightarrow{\text{horizontal } \mathcal{H}} G \Big/ (K \cdot H).$$

- The tangent bundle decomposes into $T(G/K) = \mathcal{H} \oplus \mathcal{V}$. We define the canonical variation g_{t_0,t_1} by scaling these parts:

$$\boxed{g_{t_0,t_1} := t_0 \cdot g \restriction_{\mathcal{H}} + t_1 \cdot g \restriction_{\mathcal{V}} \quad (t_0, t_1 > 0)}$$

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Problem: If $t_0 \neq t_1$, the space $(G/K, g_{t_0,t_1})$ is generally **not** naturally reductive anymore, so our main formula does not apply!

Our Trick: Finding a Naturally Reductive Realization

The Situation

For a simple group G, the space $(G/K, g_{t_0,t_1})$ is naturally reductive if and only if g is the Killing form and $t_0 = t_1$.

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Aim: Find a **new**, larger group pair (G', K') such that

$$(G/K, g_{t_0,t_1}) \cong (G'/K', h_{r_0,r_1})$$

where the new space on the right **is** naturally reductive!

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Solution: Define a new space $(G'/K', h_{r_0,r_1})$ by:

Groups: $G' = G \times H$ and $K' = \{(kh, h) \mid k \in K, h \in H\}$.

Metric: $h_{r_0,r_1} := r_0 g|_{\mathfrak{g}} + r_1 g|_{\mathfrak{h}}$ for $r_0, r_1 \in \mathbb{R}^*$.

The Main Isometry Theorem

Theorem [A-H., 2024]

The map

$$\Psi : \quad (G/K, g_{t_0,t_1}) \to (G'/K', h_{r_0,r_1}), \quad gK \mapsto (g,e)K'$$

is a G-equivariant isometry for

$$t_0 = r_0, \quad \frac{1}{r_1} = \frac{1}{t_1} - \frac{1}{t_0}.$$

This yields a family of naturally reductive realizations for the deformed spaces.

This trick, inspired by Wilking (1999), allows us to use the standard formula on the new space (G'/K').

The Spectrum of Canonical Variations

Applying the standard formula to $(G'/K', h_{r_0,r_1})$ yields the spectrum for our original deformed space:

Theorem [A-H., 2024]

The spectrum of $(G/K, g_{t_0,t_1})$ is given by

$$\Sigma(G/K, g_t) = \left\{ \frac{c_g(\varrho_0) - c_{g\restriction\mathfrak{h}}(\varrho_1)}{t_0} + \frac{c_{g\restriction\mathfrak{h}}(\varrho_1)}{t_1} \;\middle|\; \varrho_0 \otimes \varrho_1 \in \widehat{G'}_{K'} \right\}$$

The multiplicity is given by $m(\varrho_0 \otimes \varrho_1) \cdot \dim V_{\varrho_0 \otimes \varrho_1}$.

What is left to do? Determine the spherical representations $\widehat{G'}_{K'}$!

A large Family of Examples:

The Class of positive 3-(α, δ)-Sasaki mfds

"They are canonical variations of 3-Sasaki manifolds!"

Construction of 3-(α,δ)-Sasaki Manifolds

- Homogeneous 3-Sasaki manifolds $(G/K, g)$ (simply connected) are in one-to-one correspondence with compact simple Lie groups G.
- An $\mathrm{SU}(2) \subset G$ commutes with K, leading to a submersion:

$$\mathrm{SU}(2) \Big/ (\mathrm{SU}(2) \cap K) \to G/K \to G \Big/ (K \cdot \mathrm{SU}(2))$$

- **Fact (Agricola, Dileo, 2020):** Positive, homogeneous 3-(α,δ)-Sasaki manifolds are precisely the canonical variations g_{t_0,t_1} of these 3-Sasaki manifolds.
- The parameters are identified as:

$$t_0 = \frac{1}{2\alpha\delta}, \quad t_1 = \frac{1}{\delta^2} \quad (\text{for } \alpha, \delta > 0)$$

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- **Special Cases:**
  - *3-α-Sasaki metric:* $\alpha = \delta$ (i.e., $t_1 = 2t_0$).
  - *Einstein metric:* $\delta = \alpha$ or $\delta = (2n + 3)\alpha$.
  - If $n = 1$, the latter Einstein metric is due to a nearly parallel G_2-structure

Spectrum of 3-(α, δ)-Sasaki manifolds

Our general formula applies directly:

Corollary [A-H., 2024]

The spectrum of a 3-(α, δ)-Sasaki manifold is given by

$$\left\{ 2\alpha\delta\left(c_{\mathfrak{g}}(\varrho_0) - c_{\mathfrak{g}\upharpoonright\mathfrak{su}(2)}(\varrho_1)\right) + \delta^2 c_{\mathfrak{g}\upharpoonright\mathfrak{su}(2)}(\varrho_1) \mid \varrho_0 \otimes \varrho_1 \in \widehat{G \times \mathrm{SU}(2)}_{K'} \right\}$$

- Determining the branching and computing the Casimir constants leads to explicit formulas.
- The branching builds partially on the work of Chami (2004, 2012) and is joint work with Leandro Cagliero.
- We computed the spectrum for all compact, homogeneous, non-exceptional 3-(α, δ)-Sasaki manifolds.

Example: Type A_l Spectrum

Theorem

The spectrum of $(\mathrm{SU}(n+1)/S(\mathrm{U}(n-1)\times \mathrm{U}(1)), g_{t_0,t_1})$ is given by the collection of numbers:

$$2\alpha\delta\cdot\left(2\left(\sum_{k=1}^{n} z_k^2+\sum_{k=1}^{n}(n+2-2k)z_k-\frac{1}{n+1}\left(\sum_{k=1}^{n} z_k\right)^2\right)\right.$$
$$\left.-\,z_{n+1}\big(2+z_{n+1}\big)\right)+\delta^2\, z_{n+1}\big(2+z_{n+1}\big),$$

where $z_1\geq\cdots\geq z_n\geq 0$, $z_{n+1}\geq 0$ are natural numbers satisfying the conditions on the next slide. The spherical multiplicity is given by $m(\varrho_0(z_1,\dots,z_n)\otimes\varrho_1(z_{n+1}))=1$.

Type A_l: The 3-(α, δ)-Spectrum – Conditions

Conditions on z_i

The numbers $z_i \in \mathbb{N}_0$ satisfy the following conditions:

1. **Case $n = 2$:**
$$z_1 + z_2 \equiv 0 \pmod 3, \quad z_3 \equiv 0 \pmod 2,$$
$$z_1 \geq \frac{z_1 + z_2}{3} + \frac{z_3}{2} \geq z_2 \geq \frac{z_1 + z_2}{3} - \frac{z_3}{2} \geq 0.$$
2. **Case $n = 3$:**
$$z_1 + z_2 + z_3 \equiv 0 \pmod 4, \qquad z_3 \leq \frac{z_1 + z_2 + z_3}{4} \leq z_2.$$
z_4 must be even and satisfy:
$$|z_1 - z_2 - z_3| \leq z_4 \leq z_1 - z_2 + z_3.$$
3. **Case $n \geq 4$:**
$$z_1 + z_2 + z_n = 4z_i, \quad \text{for } i = 3, \ldots, n-1.$$
z_{n+1} must be even and satisfy:
$$|z_1 - z_2 - z_n| \leq z_{n+1} \leq z_1 - z_2 + z_n.$$

Our Inspiration: Wilking's Normal Homogeneous Realization of $(W^{1,1}, g_{t_0,t_1})$

"Recognizing whether a given homogeneous space is normal homogeneous can be tricky!"

The space $W^{1,1}$

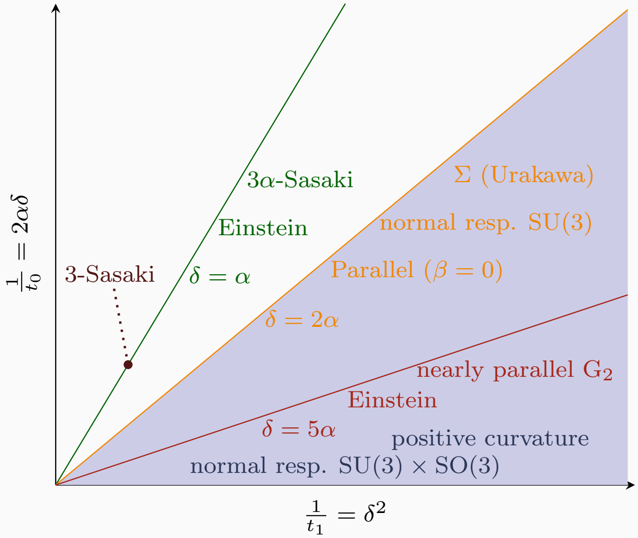
Definition

The 3-(α, δ)-Sasaki manifold $W^{1,1} := \mathrm{SU}(3) / S^1$ is called an Aloff-Wallach space.

- Urakawa (1984) computed its spectrum for the undeformed case $(t_0 = t_1)$.
- For $0 < t_1 < t_0$, the space has positive curvature.
- **Wilking (1999)** found a different, normal homogeneous realization for this case:

$$V_3 := \mathrm{SU}(3) \times \mathrm{SO}(3) \Big/ \mathrm{U}^\bullet(2)$$

⇝ This closed a gap in Berger's classification of normal homogeneous spaces with positive curvature. Wilking's construction can be obtained by our general method.



Our method covers all naturally reductive realizations of $(W^{1,1}, g_{t_0,t_1})$.

The Spectrum of $(W^{1,1}, g_{t_0,t_1})$

Theorem [A-H., 2023]

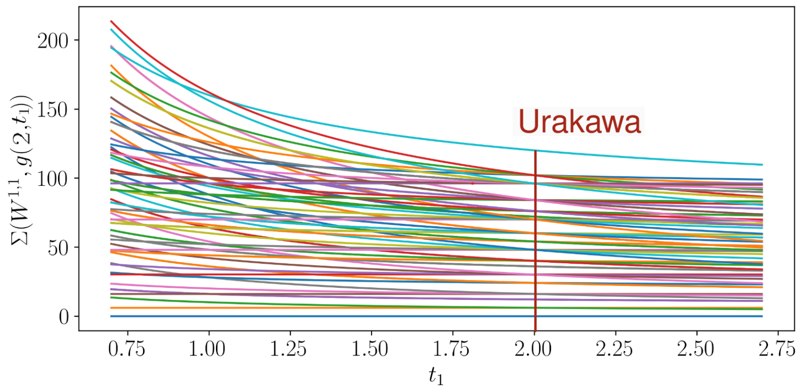
The spectrum of $(W^{1,1}, g_{t_0,t_1})$ is given by

$$\eta(z_1, z_2, z_3) = \frac{{z_3}^2 + 2z_3}{t_1} + \frac{4({z_1}^2 + {z_2}^2 - z_1(z_2 - 3)) - 3({z_3}^2 + 2z_3)}{3t_0}$$

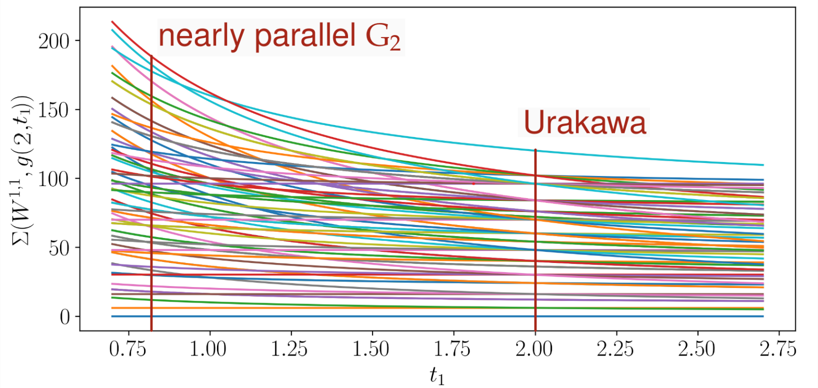
with $z_1, z_2, z_3 \in \mathbb{N}_0$ satisfying

$$z_1 + z_2 \equiv 0 \mod (3), \quad z_3 \equiv 0 \pmod 2,$$
$$z_1 \geq \frac{z_1 + z_2}{3} + \frac{z_3}{2} \geq z_2 \geq \frac{z_1 + z_2}{3} - \frac{z_3}{2} \geq 0.$$

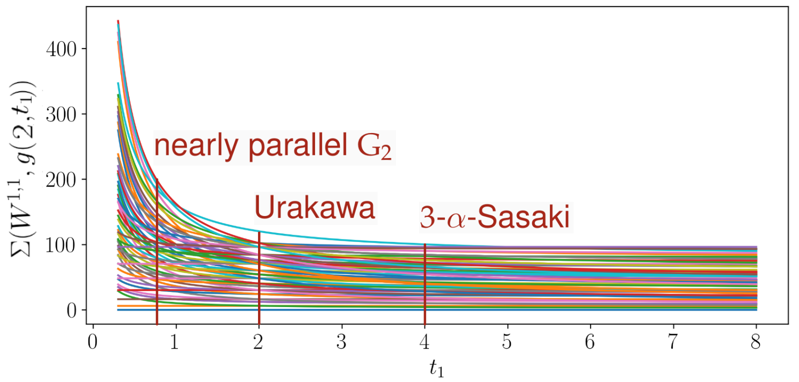
- The case $t_0 = t_1$ recovers the spectrum computed by Urakawa.

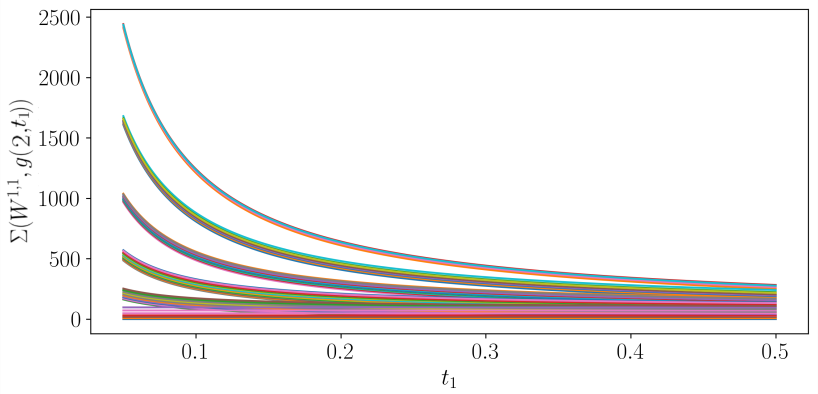


- Multiplicities of the undeformed spectrum (Urakawa, $t_1 = 2$) split under deformation.
- Some eigenvalues ($z_3 = 0$) are constant: they correspond to eigenfunctions lifted from the base space $\mathbb{CP}^2$ of the Riemannian submersion $\mathrm{SO}(3) \to W^{1,1} \to \mathbb{CP}^2$.



Special geometries like the nearly parallel G_2 case $(t_1 = 0.8)$ or the 3-α-Sasaki case $(t_1 = 4)$ seem **not** to be special points in the 1-parameter spectrum.


nearly parallel G_2
Urakawa
3-α-Sasaki
$\Sigma(W^{1,1}, g(2,t_1))$
400
300
200
100
0
0
1
2
3
4
5
6
7
8
t_1



Each "bundle" is generated by the same SU(2)-representation $\varrho(z_3)$ which yields an eigenvalue of the fibre SO(3) of the Riemannian submersion SO(3) $\rightarrow W^{1,1} \rightarrow \mathbb{C}\,\mathbb{P}^2$.