An AI-Assisted Proof in Spectral Geometry

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The Hodge Laplacian

and its first eigenvalue

Motivation

Framework:

- Let (M, g) be a closed Riemannian manifold, $\Delta_p = d\delta + \delta d$ be the Hodge-Laplacian acting on p-forms.
- Outside of (locally) symmetric spaces, explicit full Hodge spectra (where $p \neq 0$) are very rare, as they are difficult to compute.

Framework:

- Let (M, g) be a closed Riemannian manifold, $\Delta_p = d\delta + \delta d$ be the Hodge-Laplacian acting on p-forms.
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Our Specific Goal: Homogeneous 3-Spheres

- General left-invariant metrics $g_{a,b,c}$ on $S^3 \cong \mathrm{SU}(2)$.
- How does the spectrum reflect the geometry? Does it uniquely determine the isometry class?
- *Status:* Solved for functions ($p = 0$). Open for $p > 0$ (where it suffices to consider $p = 1$ due to Hodge duality).

Exploiting the symmetries: Representation theory

- By the Peter-Weyl Theorem for $S^3 \cong \mathrm{SU}(2)$, the space of 1-forms decomposes into irreducible representations $(\varrho(k), V_k)$:

$$L^2(\textstyle\bigwedge^1 T^*S^3) = \overline{\bigoplus_{\varrho(k)\in\widehat{\mathrm{SU}(2)}} V_k^* \otimes \mathfrak{su}(2)^*_{\mathbb{C}} \otimes V_k}$$

- *Consequence:* The operator Δ_1 reduces to a family of finite-dimensional matrices $\Delta_1^{(k)}$, indexed by the highest weight k.

Δ FOR $\varrho(0)$ = TRIVIAL REPRESENTATION

$$\begin{pmatrix} \frac{4b^2c^2}{a^2} & 0 & 0 \\ 0 & \frac{4a^2c^2}{b^2} & 0 \\ 0 & 0 & \frac{4a^2b^2}{c^2} \end{pmatrix}$$

Δ FOR $\varrho(1)$ = STANDARD REPRESENTATION

$$\begin{pmatrix}
a^2+b^2+\frac{4b^2c^2}{a^2}+c^2 & 0 & 0 & 0 & 2ib^2-\frac{2ib^2c^2}{a^2}-2ic^2 & 2b^2+\frac{2b^2c^2}{a^2}-2c^2 \\
0 & a^2+b^2+\frac{4a^2c^2}{b^2}+c^2 & 2ia^2+\frac{2ia^2c^2}{b^2}-2ic^2 & -2ia^2+\frac{2ia^2c^2}{b^2}+2ic^2 & 0 & 0 \\
0 & -2ia^2+2ib^2-\frac{2ia^2b^2}{c^2} & a^2+b^2+\frac{4a^2b^2}{c^2}+c^2 & 2a^2-2b^2-\frac{2a^2b^2}{c^2} & 0 & 0 \\
0 & 2ib^2-\frac{2ib^2c^2}{a^2}-2ic^2 & -2b^2-\frac{2b^2c^2}{a^2}+2c^2 & a^2+b^2+\frac{4b^2c^2}{a^2}+c^2 & 0 & 0 \\
-2ia^2+\frac{2ia^2c^2}{b^2}+2ic^2 & 0 & 0 & 0 & a^2+b^2+\frac{4a^2c^2}{b^2}+c^2 & -2ia^2-\frac{2ia^2c^2}{b^2}+2ic^2 \\
-2a^2+2b^2+\frac{2a^2b^2}{c^2} & 0 & 0 & 0 & 2ia^2-2ib^2+\frac{2ia^2b^2}{c^2} & a^2+b^2+\frac{4a^2b^2}{c^2}+c^2
\end{pmatrix}$$

Δ FOR $\varrho(2)$ = ADJOINT REPRESENTATION

$$\begin{pmatrix}
4a^2+2b^2+\frac{4b^2c^2}{a^2}+2c^2 & 0 & 0 & 0 & -2\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 2b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & -2b^2+2c^2 & 0 & 0 \\
0 & 4a^2+2b^2+\frac{4a^2c^2}{b^2}+2c^2 & 4a\left(\frac{iab}{c}+\frac{iac}{b}-\frac{ibc}{a}\right) & -2\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & 0 & 0 & 0 & -2b^2+2c^2 & 0 \\
0 & 4a\left(-\frac{iab}{c}-\frac{iac}{b}+\frac{ibc}{a}\right) & 4a^2+2b^2+\frac{4a^2b^2}{c^2}+2c^2 & -2b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & 0 & -2b^2+2c^2 \\
0 & 4\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & -4b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 4b^2+\frac{4b^2c^2}{a^2}+4c^2 & 0 & 0 & 0 & 4\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & 4b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) \\
4\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 & 0 & 4b^2+\frac{4a^2c^2}{b^2}+4c^2 & 0 & 4\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 \\
4b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & 0 & 4b^2+\frac{4a^2b^2}{c^2}+4c^2 & -4b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 \\
-2b^2+2c^2 & 0 & 0 & 0 & -2\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & -2b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 4a^2+2b^2+\frac{4b^2c^2}{a^2}+2c^2 & 0 & 0 \\
0 & -2b^2+2c^2 & 0 & -2\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & 0 & 0 & 0 & 4a^2+2b^2+\frac{4a^2c^2}{b^2}+2c^2 & 4a\left(-\frac{iab}{c}-\frac{iac}{b}+\frac{ibc}{a}\right) \\
0 & 0 & -2b^2+2c^2 & 2b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & 4a\left(\frac{iab}{c}+\frac{iac}{b}-\frac{ibc}{a}\right) & 4a^2+2b^2+\frac{4a^2b^2}{c^2}+2c^2
\end{pmatrix}$$

$$\left(\begin{array}{cccccccccccc}
9a^2+3b^2+\frac{4b^2c^2}{a^2}+3c^2 & 0 & 0 & 0 & -2\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 2b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & -2b^2+2c^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 9a^2+3b^2+\frac{4a^2c^2}{b^2}+3c^2 & 6a\left(\frac{iab}{c}+\frac{iac}{b}-\frac{ibc}{a}\right) & -2\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & 0 & 0 & 0 & -2b^2+2c^2 & 0 & 0 & 0 & 0 \\
0 & 6a\left(-\frac{iab}{c}-\frac{iac}{b}+\frac{ibc}{a}\right) & 9a^2+3b^2+\frac{4a^2b^2}{c^2}+3c^2 & -2b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & 0 & -2b^2+2c^2 & 0 & 0 & 0 \\
0 & 6\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & -6b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & a^2+7b^2+\frac{4b^2c^2}{a^2}+7c^2 & 0 & 0 & 0 & 4\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & 4b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & -6b^2+6c^2 & 0 & 0 \\
6\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 & 0 & a^2+7b^2+\frac{4a^2c^2}{b^2}+7c^2 & -2a\left(-\frac{iab}{c}-\frac{iac}{b}+\frac{ibc}{a}\right) & 4\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 & 0 & -6b^2+6c^2 & 0 \\
6b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & -2a\left(\frac{iab}{c}+\frac{iac}{b}-\frac{ibc}{a}\right) & a^2+7b^2+\frac{4a^2b^2}{c^2}+7c^2 & -4b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & 0 & -6b^2+6c^2 \\
-6b^2+6c^2 & 0 & 0 & 0 & 4\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & -4b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & a^2+7b^2+\frac{4b^2c^2}{a^2}+7c^2 & 0 & 0 & 0 & 6\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & 6b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) \\
0 & -6b^2+6c^2 & 0 & 4\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 & 0 & a^2+7b^2+\frac{4a^2c^2}{b^2}+7c^2 & -2a\left(\frac{iab}{c}+\frac{iac}{b}-\frac{ibc}{a}\right) & 6\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 \\
0 & 0 & -6b^2+6c^2 & 4b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & -2a\left(-\frac{iab}{c}-\frac{iac}{b}+\frac{ibc}{a}\right) & a^2+7b^2+\frac{4a^2b^2}{c^2}+7c^2 & -6b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 \\
0 & 0 & 0 & -2b^2+2c^2 & 0 & 0 & 0 & -2\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & -2b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 9a^2+3b^2+\frac{4b^2c^2}{a^2}+3c^2 & 0 & 0 \\
0 & 0 & 0 & 0 & -2b^2+2c^2 & 0 & -2\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & 0 & 0 & 0 & 9a^2+3b^2+\frac{4a^2c^2}{b^2}+3c^2 & 6a\left(-\frac{iab}{c}-\frac{iac}{b}+\frac{ibc}{a}\right) \\
0 & 0 & 0 & 0 & 0 & -2b^2+2c^2 & 2b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & 6a\left(\frac{iab}{c}+\frac{iac}{b}-\frac{ibc}{a}\right) & 9a^2+3b^2+\frac{4a^2b^2}{c^2}+3c^2
\end{array}\right)$$

Δ FOR $\varrho(4)$

$$
\begin{pmatrix}
16a^2+4b^2+\frac{4b^2c^2}{a^2}+4c^2 & 0 & 0 & 0 & -2\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 2b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & -2b^2+2c^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 16a^2+4b^2+\frac{4a^2c^2}{b^2}+4c^2 & 8a\left(\frac{iab}{c}+\frac{iac}{b}-\frac{ibc}{a}\right) & -2\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & 0 & 0 & 0 & -2b^2+2c^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 8a\left(-\frac{iab}{c}-\frac{iac}{b}+\frac{ibc}{a}\right) & 16a^2+4b^2+\frac{4a^2b^2}{c^2}+4c^2 & -2b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & 0 & -2b^2+2c^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 8\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & -8b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 4a^2+10b^2+\frac{4b^2c^2}{a^2}+10c^2 & 0 & 0 & 0 & 4\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & 4b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & -6b^2+6c^2 & 0 & 0 & 0 & 0 & 0 \\
8\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 & 0 & 4a^2+10b^2+\frac{4a^2c^2}{b^2}+10c^2 & 4a\left(\frac{iab}{c}+\frac{iac}{b}-\frac{ibc}{a}\right) & 4\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 & 0 & -6b^2+6c^2 & 0 & 0 & 0 & 0 \\
8b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & 4a\left(-\frac{iab}{c}-\frac{iac}{b}+\frac{ibc}{a}\right) & 4a^2+10b^2+\frac{4a^2b^2}{c^2}+10c^2 & -4b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & 0 & -6b^2+6c^2 & 0 & 0 & 0 \\
-12b^2+12c^2 & 0 & 0 & 0 & 6\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & -6b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 12b^2+\frac{4b^2c^2}{a^2}+12c^2 & 0 & 0 & 0 & 6\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & 6b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & -12b^2+12c^2 & 0 & 0 \\
0 & -12b^2+12c^2 & 0 & 6\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 & 0 & 12b^2+\frac{4a^2c^2}{b^2}+12c^2 & 0 & 6\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 & 0 & -12b^2+12c^2 & 0 \\
0 & 0 & -12b^2+12c^2 & 6b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & 0 & 12b^2+\frac{4a^2b^2}{c^2}+12c^2 & -6b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & 0 & -12b^2+12c^2 \\
0 & 0 & 0 & -6b^2+6c^2 & 0 & 0 & 0 & 4\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & -4b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 4a^2+10b^2+\frac{4b^2c^2}{a^2}+10c^2 & 0 & 0 & 0 & 8\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & 8b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) \\
0 & 0 & 0 & 0 & -6b^2+6c^2 & 0 & 4\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 & 0 & 4a^2+10b^2+\frac{4a^2c^2}{b^2}+10c^2 & 4a\left(-\frac{iab}{c}-\frac{iac}{b}+\frac{ibc}{a}\right) & 8\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -6b^2+6c^2 & 4b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & 4a\left(\frac{iab}{c}+\frac{iac}{b}-\frac{ibc}{a}\right) & 4a^2+10b^2+\frac{4a^2b^2}{c^2}+10c^2 & -8b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2b^2+2c^2 & 0 & 0 & 0 & -2\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & -2b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 16a^2+4b^2+\frac{4b^2c^2}{a^2}+4c^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2b^2+2c^2 & 0 & -2\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & 0 & 0 & 0 & 16a^2+4b^2+\frac{4a^2c^2}{b^2}+4c^2 & 8a\left(-\frac{iab}{c}-\frac{iac}{b}+\frac{ibc}{a}\right) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2b^2+2c^2 & 2b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & 8a\left(\frac{iab}{c}+\frac{iac}{b}-\frac{ibc}{a}\right) & 16a^2+4b^2+\frac{4a^2b^2}{c^2}+4c^2
\end{pmatrix}
$$

Δ FOR $\varrho(5)$

$$
\left(\begin{array}{cccccccccccccccccc}
25a^2+5b^2+\frac{4b^2c^2}{a^2}+5c^2 & 0 & 0 & 0 & -2\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 2b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & -2b^2+2c^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 25a^2+5b^2+\frac{4a^2c^2}{b^2}+5c^2 & 10a\left(\frac{iab}{c}+\frac{iac}{b}-\frac{ibc}{a}\right) & -2\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & 0 & 0 & 0 & -2b^2+2c^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 10a\left(-\frac{iab}{c}-\frac{iac}{b}+\frac{ibc}{a}\right) & 25a^2+5b^2+\frac{4a^2b^2}{c^2}+5c^2 & -2b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & 0 & -2b^2+2c^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 10\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & -10b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 9a^2+13b^2+\frac{4b^2c^2}{a^2}+13c^2 & 0 & 0 & 0 & 4\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & 4b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & -6b^2+6c^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 & 0 & 9a^2+13b^2+\frac{4a^2c^2}{b^2}+13c^2 & 6a\left(\frac{iab}{c}+\frac{iac}{b}-\frac{ibc}{a}\right) & 4\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 & 0 & -6b^2+6c^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & 6a\left(-\frac{iab}{c}-\frac{iac}{b}+\frac{ibc}{a}\right) & 9a^2+13b^2+\frac{4a^2b^2}{c^2}+13c^2 & -4b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & 0 & -6b^2+6c^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-20b^2+20c^2 & 0 & 0 & 0 & 8\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & -8b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & a^2+17b^2+\frac{4b^2c^2}{a^2}+17c^2 & 0 & 0 & 0 & 6\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & 6b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & -12b^2+12c^2 & 0 & 0 & 0 & 0 & 0 \\
0 & -20b^2+20c^2 & 0 & 8\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 & 0 & a^2+17b^2+\frac{4a^2c^2}{b^2}+17c^2 & -2a\left(-\frac{iab}{c}-\frac{iac}{b}+\frac{ibc}{a}\right) & 6\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 & 0 & -12b^2+12c^2 & 0 & 0 & 0 & 0 \\
0 & 0 & -20b^2+20c^2 & 8b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & -2a\left(\frac{iab}{c}+\frac{iac}{b}-\frac{ibc}{a}\right) & a^2+17b^2+\frac{4a^2b^2}{c^2}+17c^2 & -6b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & 0 & -12b^2+12c^2 & 0 & 0 & 0 \\
0 & 0 & 0 & -12b^2+12c^2 & 0 & 0 & 0 & 6\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & -6b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & a^2+17b^2+\frac{4b^2c^2}{a^2}+17c^2 & 0 & 0 & 0 & 8\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & 8b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & -20b^2+20c^2 & 0 & 0 \\
0 & 0 & 0 & 0 & -12b^2+12c^2 & 0 & 6\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 & 0 & a^2+17b^2+\frac{4a^2c^2}{b^2}+17c^2 & -2a\left(\frac{iab}{c}+\frac{iac}{b}-\frac{ibc}{a}\right) & 8\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 & 0 & -20b^2+20c^2 & 0 \\
0 & 0 & 0 & 0 & 0 & -12b^2+12c^2 & 6b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & -2a\left(-\frac{iab}{c}-\frac{iac}{b}+\frac{ibc}{a}\right) & a^2+17b^2+\frac{4a^2b^2}{c^2}+17c^2 & -8b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & 0 & -20b^2+20c^2 \\
0 & 0 & 0 & 0 & 0 & 0 & -6b^2+6c^2 & 0 & 0 & 0 & 4\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & -4b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 9a^2+13b^2+\frac{4b^2c^2}{a^2}+13c^2 & 0 & 0 & 0 & 10\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & 10b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -6b^2+6c^2 & 0 & 4\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 & 0 & 9a^2+13b^2+\frac{4a^2c^2}{b^2}+13c^2 & 6a\left(-\frac{iab}{c}-\frac{iac}{b}+\frac{ibc}{a}\right) & 10\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6b^2+6c^2 & 4b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 & 0 & 6a\left(\frac{iab}{c}+\frac{iac}{b}-\frac{ibc}{a}\right) & 9a^2+13b^2+\frac{4a^2b^2}{c^2}+13c^2 & -10b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2b^2+2c^2 & 0 & 0 & 0 & -2\left(-\frac{iab}{c}+\frac{iac}{b}+\frac{ibc}{a}\right)c & -2b\left(\frac{ab}{c}-\frac{ac}{b}+\frac{bc}{a}\right) & 25a^2+5b^2+\frac{4b^2c^2}{a^2}+5c^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2b^2+2c^2 & 0 & -2\left(\frac{iab}{c}-\frac{iac}{b}-\frac{ibc}{a}\right)c & 0 & 0 & 0 & 25a^2+5b^2+\frac{4a^2c^2}{b^2}+5c^2 & 10a\left(-\frac{iab}{c}-\frac{iac}{b}+\frac{ibc}{a}\right) \\
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\end{array}\right)
$$

Intermediate Success & The Bottleneck

Berger Spheres ($b = c$):

- Successfully diagonalized the matrices $\Delta_1^{(k)}$.
- The spectrum reflects the bundle curvature of the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$.

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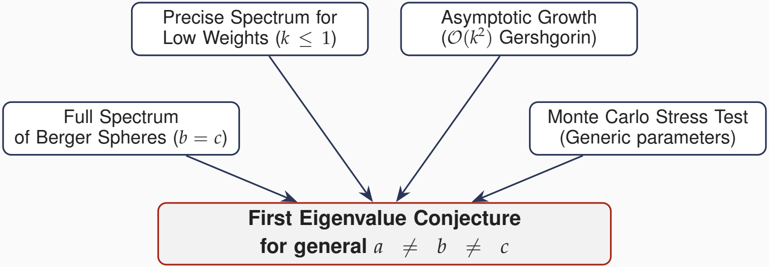
The Bottleneck: General Metrics ($a \neq b \neq c$)

- The loss of symmetry drastically increases the number of non-zero off-diagonal entries.
- Consequence: Finding the global minimum λ_1 analytically became intractable.

The Research Workflow

To bypass this analytical difficulty, we combined exact low-weight calculations with numerical methods:



Based on this workflow, we formulated the following:

Conjecture (H., Lauret 2025)

For any homogeneous metric $g_{(a,b,c)}$ on $\mathrm{SU}(2)$, the first non-zero eigenvalue of the Hodge-Laplacian on 1-forms is exactly given by:

$$\lambda_1 = \min\left\{\frac{4b^2c^2}{a^2}, \frac{4a^2c^2}{b^2}, \frac{4a^2b^2}{c^2}, a^2 + b^2 + c^2\right\}$$

The First Eigenvalue Conjecture

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Just before submitting our manuscript, we decided to run a final test using a newly released frontier LLM.

The AI Experiment

1 prompt, 100 minutes

Testing the Frontier

- **Model:** ChatGPT 5.4 Pro (Premium tier, $200 per month, limited to 15 prompts).
- **Input:** We provided the mathematical setup via *one single prompt*.
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Corollary: Δ_1-Isospectral $\Rightarrow$ isometric.

The Geometric Shortcut

Crucially, the AI bypassed the representation theory entirely.

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Step 1: The Curl Operator

Noting that $p = 1$ and $n = 3$, it introduced a self-adjoint curl operator for coexact forms:

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Step 3: Analysis
Through non-trivial estimates and the known first eigenvalue of the round sphere, it rigorously established the lower bound.

The Reality Check

Attribution and Motivation

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- It didn't cite the sources or explained its motivation for choosing this specific path.

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Conclusion: The CoT is not a transparent reasoning process. It is a stochastic search through latent space. We still do not know what literature and ideas were used to develop the argument.

Beyond the Black Box

Lessons for AI in Pure Mathematics

The Helicopter Effect

> *"AI tools are like taking a helicopter to drop you off at the site. You miss all the benefits of the journey itself. You just get right to the destination, which actually was only just a part of the value of solving these problems."*
>
> *— Terence Tao (February 2026)*

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Correct output is not enough. For AI to be a true research partner, we need complete transparency regarding the origin and logic of its ideas.

The Potential

Advanced LLMs have to potential to overcome algebraic barriers and act as good reasoning assistants in pure mathematics.

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The Challenge

To turn these "oracles" into transparent research partners, we must move beyond generating correct answers. We need systems that:

- Capture the actual discovery process.
- Ensure proper attribution of ideas.
- Explain the motivation behind mathematical steps.

Thank You

Jonas Henkel
henkelj@mathematik.uni-marburg.de

Preprint: "Hodge Laplacian on 1-forms of homogeneous 3-spheres"
J. Henkel, E. A. Lauret (Forthcoming)

Eigenvalue types of $\Delta^1|_\varrho(k)$

1. Type: Spectrum of functions, linear in a^2 and b^2:

$$\nu_{k,j} := b^2(k(k+2) - (k-2j)^2) + a^2(k-2j)^2, \quad 0 \leq j \leq k$$

2. Type: Eigenvalues of the fibre Laplacian $\Delta^0_{(S^1,g_a)}$:

$$\mu_{k,k+1} = \mu_{k,-1} := (k+2)^2 a^2$$

3.Type: Sum of terms proportional to a^2, b^2 and $1/a^2$:

$$\mu_{k,k} = \mu_{k,0} := k^2a^2 + 4kb^2 + 4\tfrac{b^4}{a^2}$$

$4^\pm$.Type: Involving $\nu_{k,j}$ and the norm of bundle curvature κ

$$\mu^\pm_{k,j} = \left(\sqrt{\nu_{k,j}+\kappa} \ \pm\ \sqrt{\kappa}\right)^2, \quad \kappa = b^4/a^2$$

They satisfy the relation $\mu^+_{k,j} > \nu_{k,j} > \mu^-_{k,j}$.

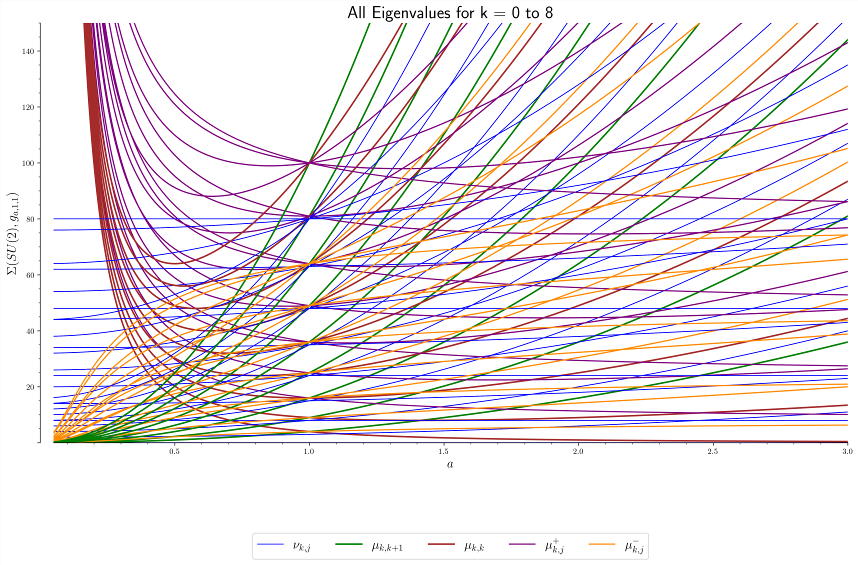

Figure 1: Full spectrum depending on a with $b = c = 1$ fixed

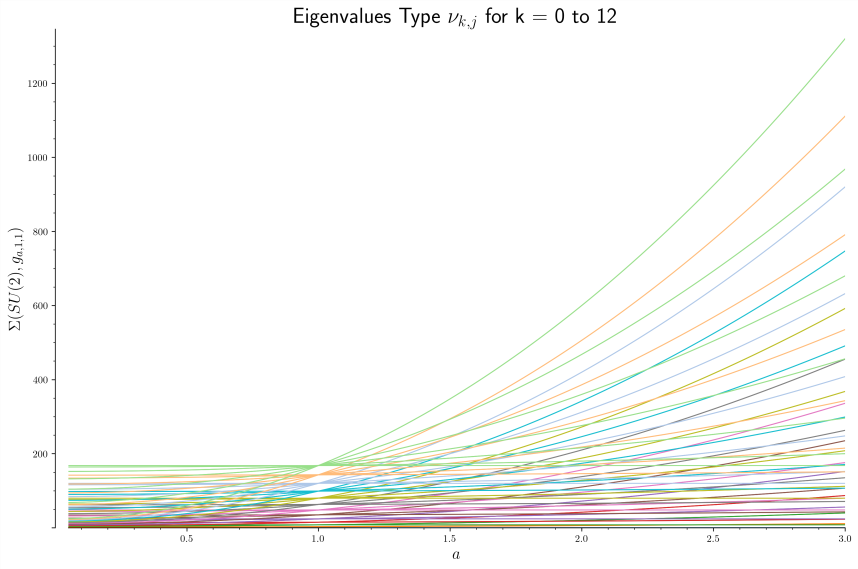

Figure 2: Type 1, known spectrum for functions

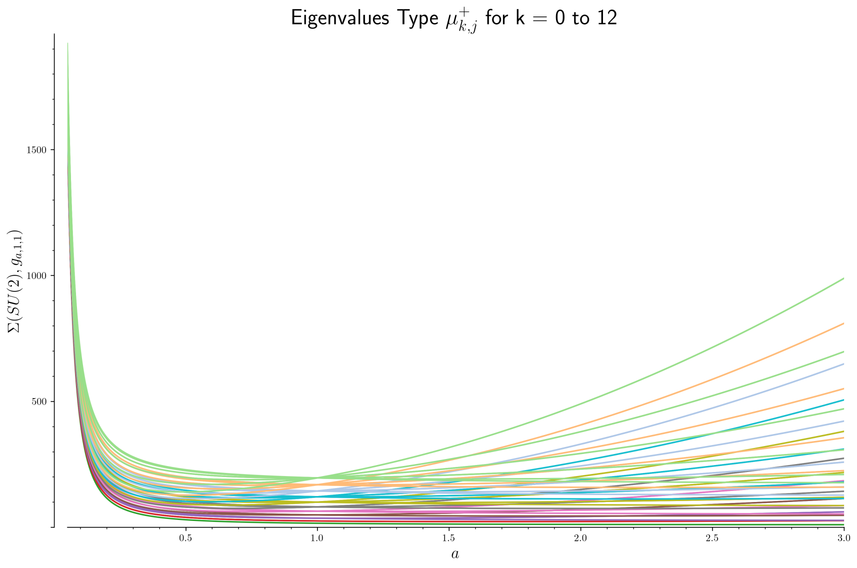

Figure 3: Type 4^{+}

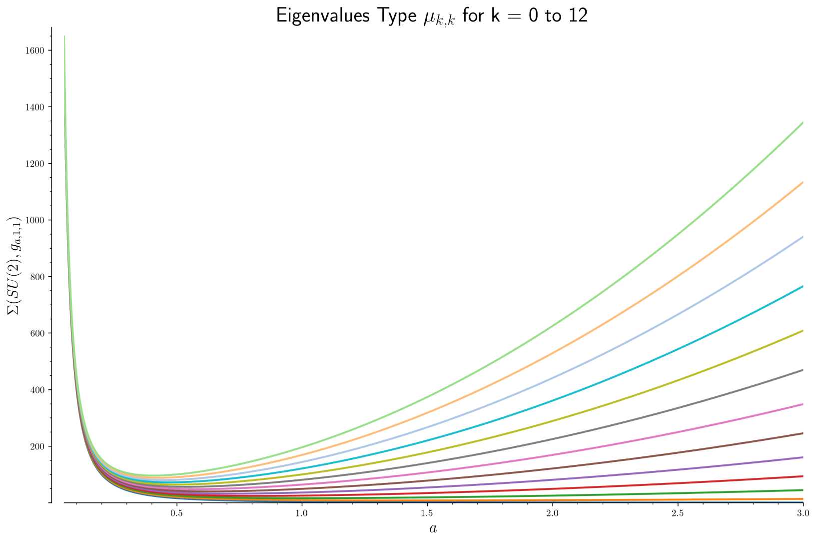

Figure 4: Type 3, $\mu_{0,0} \to 0$ for $a \to \infty$

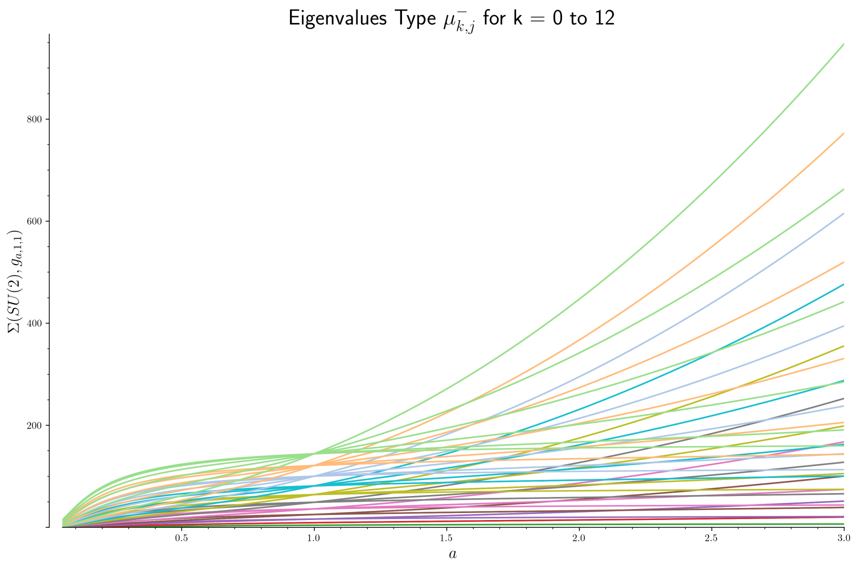

Figure 5: Type 4^-

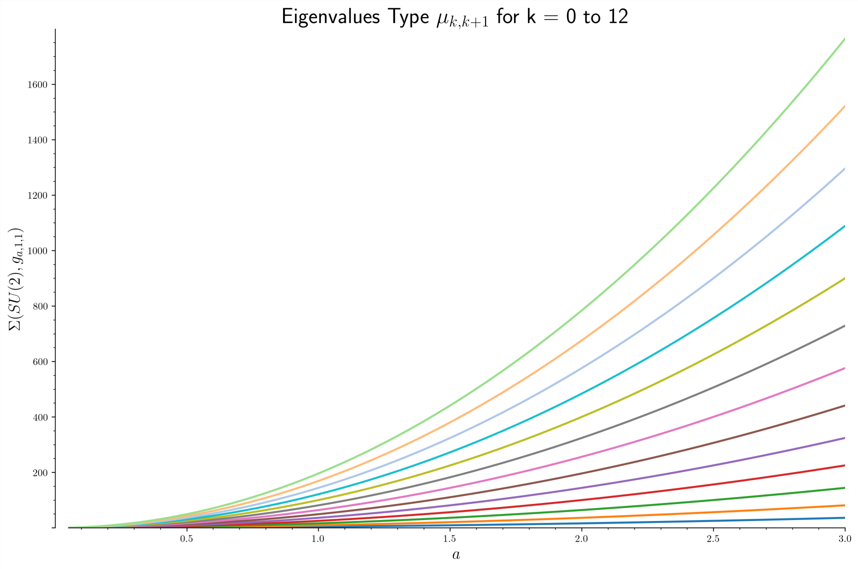

Figure 6: Type 2

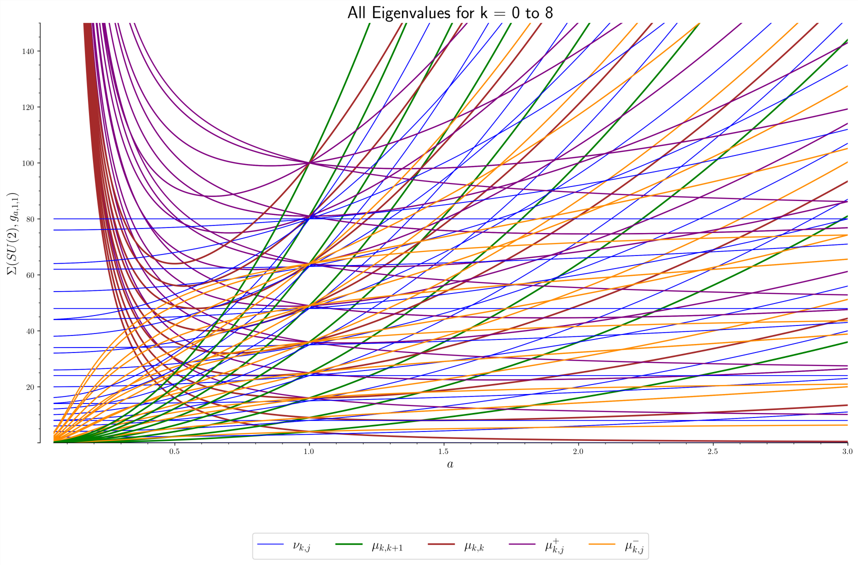

Figure 7: Full spectrum

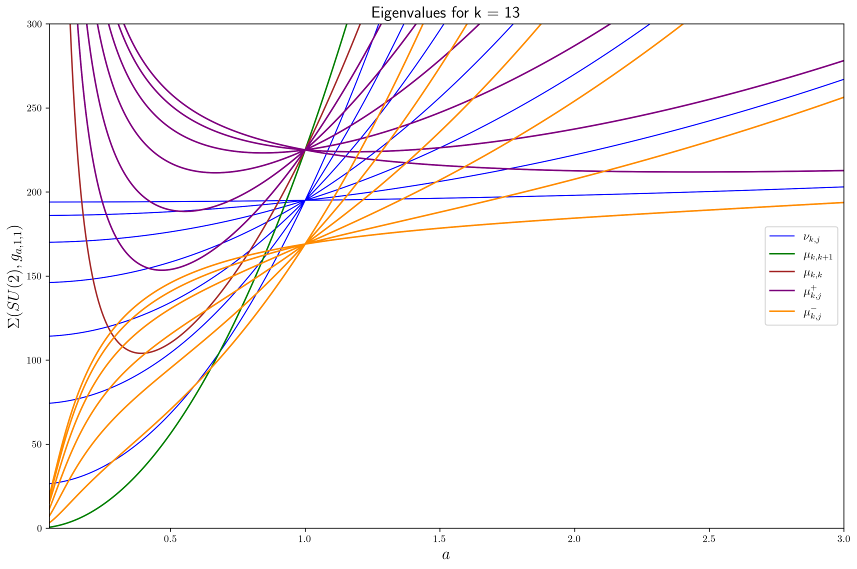

Figure 8: Spectrum. of $\Delta|_{\varrho(13)}$

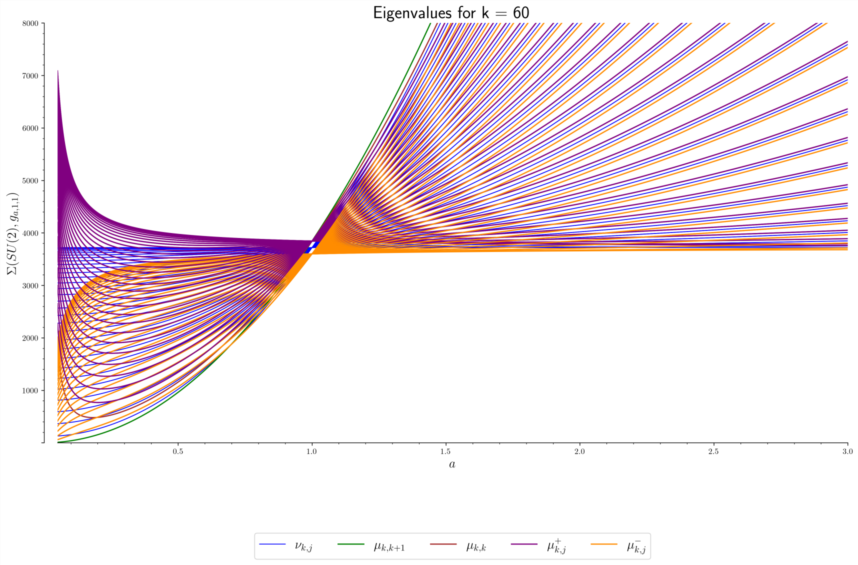

Figure 9: Spectrum of $\Delta|_{\varrho(60)}$

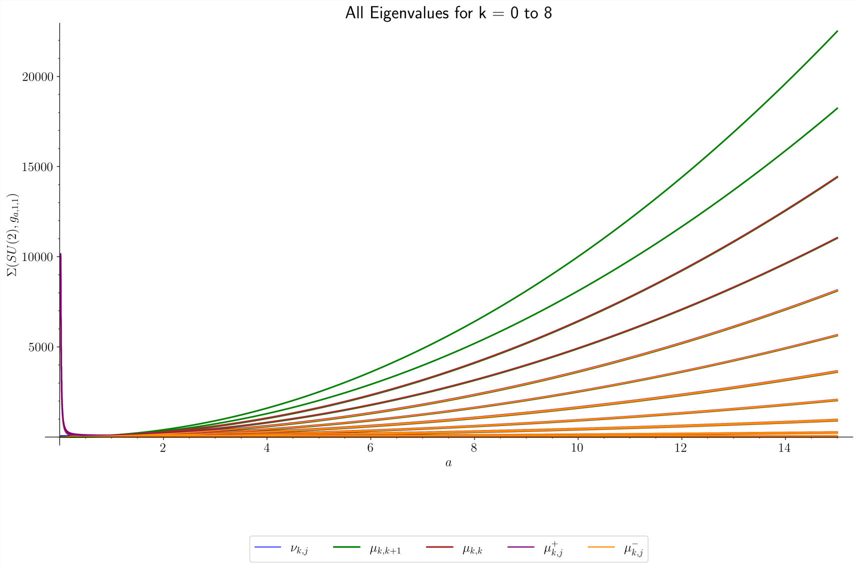

Figure 10: Full spectrum depending on a with $b = c = 1$ fixed: Bundle structure. For $a >> 0$: $\mu_{k,k+1} > \mu_{k,k} > \mu^+_{k,j} > \nu_{k,j} > \mu^-_{k,j}$, for $1 \leq j \leq k-1$